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LINEAR AND QUADRATIC INTEGRALS IN THE PROBLEM OF GYROSTAT MOTION IN A MAGNETIC FIELD†

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The necessary and sufficient conditions are established for the existence of a linear invariant relation in the problem of a gyrostat moving in a magnetic field, taking the Burnett-London effect into account. The necessary and sufficient conditions for the existence of an additional quadratic integral of the reduced system are found. © 2000 Elsevier Science Ltd. All rights reserved.

In the Hess solution [1] of the problem of a heavy rigid body moving around a fixed point, the position of the centre of mass satisfies the well-known Hess configuration condition. In Sretenskii's extension of the Hess solution [2] to the case of the motion of a heavy gyrostat, the Hess condition is also necessary for the existence of a linear invariant relation (LIR). This condition remains necessary for the existence of a LIR even in the motion of a complex mechanical system whose configuration and composition vary with time according to a given law, in a uniform gravitational field [3].

In the problem of a rigid body moving in a magnetic field, some new algebraic integrals have been obtained [4, 5]. For the motion of a gyrostat in a magnetic field, conditions have been found for the existence of a LIR of a special (Hess) type [6]. As shown in the present paper, in the presence of a magnetic moment LIRs exist of non-Hess type, including the projections of the angular momentum both onto the direction of the barycentric vector and onto the orthogonal direction in the plane of a circular section of the gyration ellipsoid.

The availability of a LIR makes it possible to reduce the order of the system. To determine integrable cases of the reduced system, it is important to solve the problem of the existence of an additional quadratic integral (QI). One example is known [6] of the existence of such a QI for a system with a LIR of a special type. In what follows, the necessary and sufficient conditions for the existence of a QI in motion with a LIR of a gyrostat in a magnetic field will be established, and the form of the QI will be determined.

1. THE CONDITIONS FOR INTEGRALS TO EXIST

We will use the following notation for the equations of motion of a gyrostat in a magnetic field [6]

$$\mathbf{x}^{\bullet} = (\mathbf{x}+\boldsymbol{\lambda})\times K\mathbf{x} + \boldsymbol{\nu}\times(C\boldsymbol{\nu} - F\mathbf{x} - \mathbf{s}), \quad \boldsymbol{\nu}^{\bullet} = \boldsymbol{\nu}\times K\mathbf{x} \tag{1.1}$$

where $\mathbf{x} = \omega$, ω is the angular velocity of the gyrostat, $\boldsymbol{\nu}$ is the unit vector in the direction of the gravity force, λ is the gyrostatic moment, s is the radius vector of the gyrostat's centre of mass and A is the inertia operator of the gyrostat at the fixed point, $K = A^{-1}$ and $F = BK$. The operators A, B and C are symmetric.

System (1.1) has first integrals

$$\boldsymbol{\nu}^2 = \text{const}, \quad (\mathbf{x}+\boldsymbol{\lambda})\boldsymbol{\nu} = \text{const} \tag{1.2}$$

A LIR of system (1.1) in general form may be written as

$$(\mathbf{p}, \mathbf{x}) + (\mathbf{q}, \boldsymbol{\nu}) = \alpha \tag{1.3}$$

In the solutions in [1–3, 6], the vectors **p** and s are collinear (**p** || s) and the coordinates of the centre

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of mass satisfy the Hess condition (2.3). When there is no magnetic moment ($B = 0$), the condition $\mathbf{p} \parallel \mathbf{s}$ is necessary. When there is a magnetic moment ($B \neq 0$), as shown below, there is a LIR of non-Hess type (the vectors $\mathbf{p}$ and $\mathbf{s}$ are not collinear).

We will formulate the result obtained for the case in which there is no dynamic symmetry. Let $A_1 > A_2 > A_3$ and let $\mathbf{e}_i$ be an eigenvector of A belonging to A_i. Along with the principal basis $\{\mathbf{e}_i\}$ we will consider a right orthobasis $\Gamma = \{\mathbf{m}_i\}$ such that $\mathbf{m}_2 = \mathbf{e}_3$ and $\mathbf{m}_1$ is a unit vector perpendicular to a circular section of the gravitational ellipsoid. If we let k_{ij} denote the elements of the matric of the operator K in the basis Γ, this basis is given by the conditions $\mathbf{m}_2 = \mathbf{e}_3$, $k_{11} = k_{22}$. Throughout what follows, a_i, $= (\mathbf{a}, \mathbf{m}_i)$.

Theorem 1. System (1.1) has a LIR if and only if $\mathbf{e}_2$ is an eigenvector of the operators B and C, the operator C has the form

$$C = c_{22}E + (c_{33} - c_{22})\mathbf{m}_3\mathbf{m}_3^T + \delta(\mathbf{q}\mathbf{m}_1^T + q_3\mathbf{m}_1\mathbf{m}_3^T) \tag{1.4}$$

and the following relations hold

$$\lambda_1 = \alpha k_{13}k_{11}^{-1}, \quad \lambda_2 = 0 \tag{1.5}$$

$$s_1 = \alpha\delta, \quad s_2 = 0 \tag{1.6}$$

$$f_{11} = q_0(k_{11}^2 - k_{13}^2), \quad f_{22} = q_0(k_{13}^2 + k_{11}^2) \tag{1.7}$$

The LIR in question may be written as

$$x_3 + q_0(k_{11}\nu_3 + k_{13}\nu_1) = k_{11}k_{13}^{-1}\lambda_1 \tag{1.8}$$

where c_{ij} and f_{ij} are the components of the operators C and F in Γ, E is the identity operator and

$$\mathbf{q} = q_0(\mathbf{m}_3\mathbf{m}_1^T + \mathbf{m}_1\mathbf{m}_3^T)K\mathbf{m}_1 \tag{1.9}$$

$$\delta = q_0k_{13}(k_{11} - k_{13}) - f_{13} \tag{1.10}$$

Note that the operator C is defined up to a term kE and the term with $c_{22}E$ in (1.4) is unimportant.

In the LIR (1.8) obtained for the general case, $p_1 = p_2 = 0$ and the case $\mathbf{p} \parallel \mathbf{s}$, considered previously in [6], is feasible only if $s_1 = 0$ which, by (1.6), is equivalent to $\alpha\delta = 0$. Thus, a LIR of the form

$$(\mathbf{s}, \mathbf{x}) + (\mathbf{q}, \boldsymbol{\nu}) = \alpha \tag{1.11}$$

may exist in two cases [6]:

1) $\alpha = 0$, $\mathbf{s} \parallel \lambda$, $\lambda \parallel \mathbf{m}_3$ and conditions (1.4) and (1.7) are satisfied;

2) $\delta = 0$, $\mathbf{s} \parallel \mathbf{m}_3$, $C = c_{22}E + (c_{33} - c_{22})\mathbf{m}_3\mathbf{m}_3^T$, $f_{13} = \langle \mathbf{q}, K\mathbf{m}_3, \mathbf{m}_2 \rangle$ and conditions (1.7) are satisfied.

The form of the LIR (1.8) is identical with that obtained in [6] for the special case mentioned. The essential difference is that, in the general case, $\mathbf{s}$ is not one of the axes of the basis Γ and it does not satisfy the Hess condition (2.3) presented below.

For motions with LIR, the order of system (1.1) may be reduced in an obvious way, by eliminating x_3 using equality (1.8). Put

$$\mathbf{z} = \mathbf{x} - x_3\mathbf{m}_3 \tag{1.12}$$

Then, since $\mathbf{p} = \mathbf{m}_3$, it follows from (1.3) that $\mathbf{x} = \mathbf{z} + [\alpha - (\mathbf{q}, \boldsymbol{\nu})]\mathbf{m}_3$, and the system of reduced order may be written as in the form

$$\begin{aligned} \mathbf{z}^\bullet &= (\mathbf{q}, \boldsymbol{\nu}^\bullet)\mathbf{m}_3 + [\mathbf{z} - (\mathbf{q}, \boldsymbol{\nu})\mathbf{m}_3 + \alpha\mathbf{m}_3 + \boldsymbol{\lambda}] \times K[\mathbf{z} - (\mathbf{q}, \boldsymbol{\nu})\mathbf{m}_3 + \alpha\mathbf{m}_3] + \\ &+ \boldsymbol{\nu} \times (C\boldsymbol{\nu} - \mathbf{s}) - \boldsymbol{\nu} \times F[\mathbf{z} - (\mathbf{q}, \boldsymbol{\nu})\mathbf{m}_3 + \alpha\mathbf{m}_3] \\ \boldsymbol{\nu}^\bullet &= \boldsymbol{\nu} \times K[\mathbf{z} - (\mathbf{q}, \boldsymbol{\nu})\mathbf{m}_3 + \alpha\mathbf{m}_3] \end{aligned} \tag{1.13}$$

Any QI of this system may be written in the form

$$(\mathbf{z},\ H_1\mathbf{z})+(\boldsymbol{\nu},\ H_2\boldsymbol{\nu})+2(H_3\,\mathbf{z},\boldsymbol{\nu})+2(\mathbf{h}_1,\ \mathbf{z})+2(\mathbf{h}_2,\boldsymbol{\nu})=\text{const} \tag{1.14}$$

The necessary and sufficient conditions for such an integral to exist are given by the following theorem.

Theorem 2. If a QI of the reduced system exists, other than integrals (1.2), it may be written in the form

$$(\mathbf{z}+\boldsymbol{\lambda})^2+(c_{33}-c_{22})k_{11}^{-1}\nu_3^2-2(k_{11}k_{13})^{-1}(\mathbf{s},\ K\mathbf{m}_1)\nu_3 \tag{1.15}$$

System (1.1) has a LIR (1.3) and (simultaneously) system (1.13) has a QI (1.14) if and only if conditions (1.5) and (1.6) hold and the operators B and C are expressible in the form

$$\begin{aligned}&B=\delta(k_{13}d)^{-1}\mathbf{b}\mathbf{b}^T,\quad C=c_{22}E+(c_{33}-c_{22})\mathbf{m}_3\mathbf{m}_3^T\\&\mathbf{b}=(K\mathbf{m}_1)\times\mathbf{m}_2,\quad d=k_{11}k_{33}-k_{13}^2\end{aligned} \tag{1.16}$$

The LIR is then $x_3=\alpha$.

Conditions for the existence of a QI in the case $\lambda=0$, corresponding to the motion of a rigid body, were considered in [6] on the assumption that a LIR of the special form (1.11) exists.

2. PROOF OF THEOREM 1

Proposition 1. No LIR (1.3) exists with $\mathbf{p}=0$.

Proof. Let the LIR have the form $(\mathbf{q},\boldsymbol{\nu})=\alpha$. Then $\boldsymbol{\nu}=\alpha\mathrm{q}|\mathrm{q}|^{-2}+\mathbf{m}$, where $(\mathbf{m},\mathbf{q})=0$. Differentiating the LIR along trajectories of system (1.1), we obtain $\langle\mathbf{q},\boldsymbol{\nu},K\mathbf{x}\rangle=0$, which is equivalent to the condition $\langle\mathbf{q},\mathbf{m},K\mathbf{x}\rangle=0\ \forall\mathbf{m}:(\mathbf{q},\mathbf{m})=0$ or $\mathbf{q}\times\mathbf{m}=0\ \forall\mathbf{m}:(\mathbf{q},\mathbf{m})=0$, so that $\mathbf{q}=0$.

Since $\mathbf{p}\neq 0$, we may assume in (1.3), without loss of generality, that $|\mathbf{p}|=1$. Any vector $\mathbf{x}$ satisfying Eq. (1.3) may be expressed in the form

$$\mathbf{x}=[\alpha-(\mathbf{q},\boldsymbol{\nu})]\mathbf{p}+\mathbf{m},\quad(\mathbf{p},\ \mathbf{m})=0 \tag{2.1}$$

Differentiating Eq. (1.3) along trajectories of system (1.1), we obtain the condition

$$\langle\mathbf{p},\mathbf{x}+\boldsymbol{\lambda},K\mathbf{x}\rangle+\langle\mathbf{p},\boldsymbol{\nu},C\boldsymbol{\nu}-F\mathbf{x}-\mathbf{s}\rangle+\langle\mathbf{q},\boldsymbol{\nu},K\mathbf{x}\rangle=0 \tag{2.2}$$

which must hold for all $\boldsymbol{\nu}$ and $\mathbf{x}$ expressible in the form of (2.1).

Proposition 2. The vector $\mathbf{p}$ in the LIR (1.3) is orthogonal to a circular section of the gravitational ellipsoid.

Proof. Substituting (2.1) into condition (2.2) and separating out the terms with m^2, we obtain the condition $\langle\mathbf{p},m,K\mathbf{x}\rangle=0\,\forall\,\mathbf{m}:(\mathbf{p},\mathbf{m})=0$, which holds if and only if

$$p^{(2)}=0,\quad a_1(p^{(1)})^2=a_3(p^{(3)})^2 \tag{2.3}$$

where $p^{(i)}$ are the coordinates of $\mathbf{p}$ in the principal basis and $a_i=A_i\Delta A_i$, $\Delta A_i=(A_j-A_k)\delta_{ijk}$, where (i, j, k) is a permutation of (1, 2, 3).

If we now introduce an orthobasis $\Gamma=\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ so that $\mathbf{m}_1=\mathbf{e}_2\times\mathbf{p}$, $\mathbf{m}_2=\mathbf{e}_2$, $\mathbf{m}_3=\mathrm{p}$, it follows from (2.3) that $k_{11}=k_{22}$, which proves Proposition 2.

Condition (2.3) implied that, if φ denotes the angle between $\mathbf{p}$ and $\mathbf{e}_1$, then $\mathrm{tg}^2\varphi=a_1/a_3$.

Proposition 3. Conditions (1.5) are necessary and sufficient for a LIR to exist.

Proof. using (2.1) to separate out the terms linear in $\mathbf{m}$ in (2.2), we obtain $\alpha\langle\mathbf{m}_3,\mathbf{m},K\mathbf{m}_3\rangle+\langle\mathbf{m}_3,\boldsymbol{\lambda},K\mathbf{m}\rangle=0$, $\mathbf{m}=m_1\mathbf{m}_1+m_2\mathbf{m}_2$ or $k_{11}\lambda_2m_1-(k_{22}\lambda_1-\alpha k_{13})m_2=0$, which, since m_1, m_2 are arbitrary, yields (1.5).

Note that the free term in condition (2.2) has the form $\alpha\langle \mathbf{m}_3, \lambda, K\mathbf{m}_3\rangle$, and it vanishes when $\lambda_2 = 0$. There remain the terms with c ν^2, νm, ν, giving the following conditions.

$$\langle \mathbf{m}_3, \boldsymbol{\nu}, C\boldsymbol{\nu} + (\mathbf{q}, \boldsymbol{\nu})F\mathbf{m}_3\rangle - (\mathbf{q}, \boldsymbol{\nu})\langle \mathbf{q}, \boldsymbol{\nu}, K\mathbf{m}_3\rangle \equiv 0 \tag{2.4}$$

$$(\mathbf{q}, \boldsymbol{\nu})\langle \mathbf{m}_3, \mathbf{m}, K\mathbf{m}_3\rangle + \langle \mathbf{m}_3, \boldsymbol{\nu}, F\mathbf{m}\rangle - \langle \mathbf{q}, \boldsymbol{\nu}, K\mathbf{m}\rangle = 0 \,\forall \boldsymbol{\nu}, \ \mathbf{m} : m_3 = 0 \tag{2.5}$$

$$(\mathbf{q}, \boldsymbol{\nu})\langle \mathbf{m}_3, \boldsymbol{\lambda}, K\mathbf{m}_3\rangle + \langle \mathbf{m}_3, \boldsymbol{\nu}, \mathbf{s} + \alpha F\mathbf{m}_3\rangle - \alpha\langle \mathbf{q}, \boldsymbol{\nu}, K\mathbf{m}_3\rangle \equiv 0 \tag{2.6}$$

Proposition 4. If a LIR exists, then necessarily $\mathbf{e}_2$ is an eigenvector of B, the parameter $\mathbf{q}$ has the form (1.9) and conditions (1.7) hold.

Proof. Setting $\mathbf{m} = \mathbf{m}_1$ and $\mathbf{m} = \mathbf{m}_2$ in condition (2.5), we obtain $\langle \mathbf{m}_3, \boldsymbol{\nu}, F\mathbf{e}_2\rangle - (\mathbf{q}, \boldsymbol{\nu})k_{13} \equiv k_{11}\langle \mathbf{q}, \boldsymbol{\nu}, \mathbf{e}_2\rangle$, $\langle \mathbf{m}_3, \boldsymbol{\nu}, F\mathbf{m}_1\rangle \equiv \langle \mathbf{q}, \boldsymbol{\nu}, K\mathbf{m}_1\rangle$. These identities hold if and only if

$$q_2 = 0, \ (\mathbf{e}_2, F\mathbf{m}_1) = 0, \ (\mathbf{m}_1, F\mathbf{e}_2) = 0$$
$$f_{11} = q_3k_{11} - q_1k_{13}, \ \ f_{22} = q_3k_{11} + q_1k_{13}, \ \ k_{11}q_1 = k_{13}q_3$$

Hence it follows that $q_1 = q_0k_{13}$, $q_3 = q_0k_{11}$. This enables us to write $\mathbf{q}$ in the form (1.9), and f_{11} and f_{22} may be written in the form (1.7). The condition $(\mathbf{e}_2, F\mathbf{m}_1) = 0$ implies, thanks to the symmetry of the operator B, that $(B\mathbf{e}_2, K\mathbf{m}_1) = 0$, and from $(\mathbf{m}_1, F\mathbf{e}_2) = 0$ it follows that $(B\mathbf{m}_1, \mathbf{e}_2) = 0$, in which case $(B\mathbf{e}_2, \mathbf{m}_1) = 0$. Since the vectors $\mathbf{m}_1, K\mathbf{m}_1$ form a basis of the plane $(\mathbf{m}_1, \mathbf{m}_3)$, it follows that $B\mathbf{e}_2 \| \mathbf{e}_2$, and $\mathbf{e}_2$ is an eigenvector of the operators B, F and F^T.

Proposition 5. Conditions (1.6) are necessary for a LIR to exist.

Proof. Condition (2.6) may be written in the form $\mathbf{m}_3 \times (\mathbf{s} + \alpha F\mathbf{m}_3) = \alpha\mathbf{q} \times K\mathbf{m}_3$, which is equivalent to conditions (1.6) with δ in the form of (1.10).

Proposition 6. If a LIR exists, the operator C must have the form of (1.4).

Proof. Setting $\boldsymbol{\nu} = \mathbf{m}_1$ in (2.4), we get $(C\mathbf{e}_2, \mathbf{m}_1) = 0$, while if $\boldsymbol{\nu} = \mathbf{m}_1 + \mathbf{m}_3$, then $(C\mathbf{e}_2, \mathbf{m}_3) = 0$. Consequently, $\mathbf{e}_2$ is an eigenvector of C. Writing identity (2.4) in the form

$$\nu_2[c_{13}\nu_3 + (c_{11} - c_{22})\nu_1] \equiv \nu_2(\nu_1q_1 + \nu_3q_3)(q_3k_{13} - q_1k_{33} - f_{13})$$

we get

$$c_{13} = q_3\delta, \ \ c_{11} - c_{22} = q_1\delta, \ \ \delta = q_3k_{13} - q_1k_{33} - f_{13}$$

This enables us to express the operator C in the form (1.4).

We have thus proved that each of the conditions in Theorem 1 is necessary for a LIR to exist. The combination of all these conditions is sufficient for a LIR to exist, since by Propositions 2–5 condition (2.2) will then hold for all $\boldsymbol{\nu}$ and all $\mathbf{x}$ given by condition (2.1). The LIR (1.3) itself may be written in the form (1.8) if we take (1.9) into consideration and use the first condition of (1.5) to express α in terms of λ_1.

3. PROOF OF THEOREM 2

Differentiating equality (1.14) along trajectories of system (1.13), we obtain the identity

$$(\mathbf{z}^{\bullet}, H_1\mathbf{z} + H_3^T\boldsymbol{\nu} + \mathbf{h}_1) + (\boldsymbol{\nu}^{\bullet}, H_2\boldsymbol{\nu} + H_3\mathbf{z} + \mathbf{h}_2) \equiv 0 \tag{3.1}$$

where $\mathbf{z}^{\bullet}$, $\boldsymbol{\nu}^{\bullet}$ are given by system (1.13).

Proposition 7. The following conditions hold

$$H_1 = \chi E, \ \ \mathbf{h}_1 = \chi\lambda_1\mathbf{m}_1 \tag{3.2}$$

and these conditions are sufficient for the terms $z^k\nu^0$ ($k = 1, 2, 3$) in (3.1) to vanish.

Proof. Setting $\nu = 0$ in identity (3.1), we obtain

$$\langle H_1\mathbf{z}+\mathbf{h}_1,\ \mathbf{z}+\alpha\mathbf{m}_3+\boldsymbol{\lambda},\ K\mathbf{z}+\alpha K\mathbf{m}_3\rangle=0,\quad \forall\mathbf{z}: z_3=0 \tag{3.3}$$

Since when $z_3 = 0$ we have $\mathbf{z} \times K\mathbf{z} = k_{13}z_1\mathbf{z} \times \mathbf{m}_3$, it follows that the condition $\langle H_1\mathbf{z}, \mathbf{z}, K\mathbf{z}\rangle = 0$, which is deduced from (3.4), may be written in the form $(H_1\mathbf{z},\ z_1\mathbf{m}_2 - z_2\mathbf{m}_1) = 0$, whence we have $\chi_{22}^{(1)} = 0$. $\chi_{22}^{(1)} = \chi_{11}^{(1)}$. Here $H_1 = \{\chi_{ij}^{(1)}\}$ in Γ. Since the elements $\chi_{3j}^{(1)}$ do not affect the form of integral (1.14), they may be assumed to be arbitrary and the operator H_1 may be written in the form of (3.2).

The term with z^2 in (3.3) takes the form $\langle \mathbf{h}_1 - \chi(\alpha\mathbf{m}_3 + \lambda), \mathbf{z}, K\mathbf{z}\rangle$, which yields the condition $z_1\langle \mathbf{h}_1 - \chi\lambda, \mathbf{m}_3, \mathbf{z}\rangle = 0$. This is possible only if $\mathbf{h}_1 - \chi\lambda \parallel \mathbf{m}_3$. In integral (1.14), $z_3 = 0$ and the quantity $(\mathbf{h}_1)_3$ is arbitrary. Putting $(\mathbf{h}_1, \mathbf{m}_3) = 0$, we obtain $\mathbf{h}_1 - \chi\lambda = -\chi\lambda_3\mathbf{m}_3$, which implies the representation (3.2) for $\mathbf{h}_1$.

We now verify that conditions (3.2) are sufficient for condition (3.3) to hold. The summand with z^0 has the form $\langle \mathbf{h}_1, \alpha\mathbf{m}_3 + \lambda, \alpha K\mathbf{m}_3\rangle$ and it vanishes because the two factors are coplanar. The terms with z^1 may be written in the form $\chi(\mathbf{a}, \mathbf{z})$, where

$$\begin{aligned}\mathbf{a} &= \alpha(\alpha\mathbf{m}_3+\boldsymbol{\lambda})\times K\mathbf{m}_3+\alpha\lambda_1 K\mathbf{m}_3\times\mathbf{m}_1+\lambda_1 K[\mathbf{m}_1\times(\alpha\mathbf{m}_3+\boldsymbol{\lambda})]=\\ &=(\alpha+\lambda_3)(\alpha k_{13}-\lambda_1 k_{22})\mathbf{m}_2\end{aligned}$$

Taking condition (1.5) and the equality $k_{11} = k_{22}$ into consideration, we obtain $\mathbf{a} = 0$.

Proposition 8. The operator H_3 may be written in the form

$$H_3=\chi n\mathbf{m}_3\mathbf{m}_1^T,\quad n=-f_{31}k_{11}^{-1}=-2q_0k_{13} \tag{3.4}$$

If $\chi \neq 0$, then necessarily

$$k_{13}f_{31}=k_{11}(f_{22}-f_{11}) \tag{3.5}$$

Proof. The terms with $z^2\nu$ in (3.1) are

$$k_{13}z_1\langle\mathbf{z},\mathbf{m}_3,H_3^T\boldsymbol{\nu}\rangle+\langle\boldsymbol{\nu},K\mathbf{z},H_3\mathbf{z}\rangle+\chi\langle\boldsymbol{\nu},\mathbf{z},F\mathbf{z}\rangle=0 \tag{3.6}$$

Setting $\mathbf{z} = \mathbf{m}_2$, we obtain $\langle\boldsymbol{\nu}, \mathbf{m}_2, H_3\mathbf{m}_2\rangle = 0$, and $\mathbf{m}_2$ is an eigenvector of the operator H_3. Putting $H_3 = \{\chi_{ij}^{(3)}\}$ in Γ, we can write

$$H_3\mathbf{z}=\chi_{22}^{(3)}\mathbf{z}+z_1\mathbf{n},\quad \mathbf{n}=H_3\mathbf{m}_1-\chi_{22}^{(3)}\mathbf{m}_1$$

It follows from integral (1.2) that

$$(\mathbf{z},\boldsymbol{\nu}) = \text{const} - (\alpha\mathbf{m}_3+\boldsymbol{\lambda})\boldsymbol{\nu} + (\mathbf{q},\boldsymbol{\nu})\nu_3$$

The term $(H_3\mathbf{z}, \boldsymbol{\nu})$ in integral (1.14) may now be written in the form

$$(H_3\mathbf{z},\boldsymbol{\nu})=\chi_{22}^{(3)}(\mathbf{z},\boldsymbol{\nu})+z_1(\mathbf{n},\boldsymbol{\nu})=z_1(\mathbf{n},\boldsymbol{\nu})+P_2(\boldsymbol{\nu})$$

where $P_2(\boldsymbol{\nu})$ is a quadratic polynomial in νi, which may be included in the terms with H_2 and $\mathbf{h}_2$ in the QI (1.14). Consequently, we may assume that $(H_3\mathbf{z}, \boldsymbol{\nu}) = z_1(\mathbf{n}, \boldsymbol{\nu})$, whence it follows that $H_3 = \mathbf{n}\mathbf{m}_1^T$. Condition (3.6) becomes

$$k_{13}z_1z_2(\mathbf{n},\boldsymbol{\nu})+z_1\langle\boldsymbol{\nu},K\mathbf{z},\mathbf{n}\rangle+\chi\langle\boldsymbol{\nu},\mathbf{z},F\mathbf{z}\rangle=0$$

This condition is satisfied only if

$$n_2=0,\quad k_{13}n_1-k_{11}n_3=\chi f_{31},\quad k_{13}n_1+k_{22}n_3=-\chi f_{31},\quad k_{13}n_3-k_{22}n_1=\chi(f_{11}-f_{22})$$

Hence it follows that $n_1 = 0$, $n_3 = -\chi k_{11}^{-1}f_{31}$, and if $\chi \neq 0$ we have relation (3.5).

Consequently, H_3 has the form (3.4). Using condition (1.7) for a LIR to exist, we infer from (3.5) that $\chi k_{11}^{-1}f_{31} = 2q_0k_{13}\chi$.

Proposition 9. If $\chi = 0$, system (1.1) has only the trivial QI $v^2 = \text{const}$.

Proof. If $\chi = 0$, identity (3.1) becomes

$$\langle \boldsymbol{\nu}, K[\mathbf{z} + \alpha \mathbf{m}_3 - (\mathbf{q}, \boldsymbol{\nu})\mathbf{m}_3], H_2\boldsymbol{\nu} + \mathbf{h}_2 \rangle = 0$$

Hence it follows that $(\boldsymbol{\nu}, K\mathbf{z}, H_2\boldsymbol{\nu}) = 0$. This is possible only if $H_2 = \chi_1 E$. Separating out the terms with vz from the identity, we obtain $(\boldsymbol{\nu}, K\mathbf{z}, \mathbf{h}2) = 0$ and $\mathbf{h}_2 = 0$. Integral (1.14) takes the form $v^2 = \text{const}$.

Supposing now that $\chi \neq 0$, we define

$$H_2 = \chi H_2', \quad \mathbf{h}_2 = \chi \mathbf{h}_2'$$

Putting

$$\mathbf{f}_1 = \mathbf{z} + n v_3 \mathbf{m}_1 + \lambda_1 \mathbf{m}_1, \quad \mathbf{f}_2 = \mathbf{z} - (\mathbf{q}, \boldsymbol{\nu})\mathbf{m}_3 + \alpha \mathbf{m}_3$$

we can write identity (3.1) as

$$\langle \mathbf{f}_1, \mathbf{f}_2, K\mathbf{f}_2 \rangle + \langle \mathbf{f}_1, \boldsymbol{\nu}, C\boldsymbol{\nu} - \mathbf{s} - F\mathbf{f}_2 \rangle + \langle \boldsymbol{\nu}, K\mathbf{f}_2, H_2'\boldsymbol{\nu} + n z_1 \mathbf{m}_3 + \mathbf{h}_2' \rangle = 0 \tag{3.7}$$

Proposition 10. A necessary condition for a QI of system (1.13) to exist other than (1.2) is that the operators B and C be expressible in the form (1.16) and that $q = n = 0$. The operator F then has the form

$$F = \delta k_{13}^{-1} \mathbf{b} \mathbf{m}_3^T \tag{3.8}$$

Proof. Separating out the terms with v^3 on the left of equality (3.7), we see, using (1.4), that

$$n v_3 v_2 [(c_{33} - c_{22}) v_3 + \delta q_3 v_1 + (\mathbf{q}, \boldsymbol{\nu}) f_{33}] \equiv (\mathbf{q}, \boldsymbol{\nu}) \langle \boldsymbol{\nu}, K\mathbf{m}_3, H_2'\boldsymbol{\nu} \rangle$$

Suppose that $n \neq 0$. Setting $v_1 = q_3$, $v_3 = -q_1$, we get $(c_{33} - c_{22}) q_1 = \delta q_3^2$, so that we can write representation (1.4) in the form $C = c_{22}E + \delta q_1^{-1} \mathbf{q}\mathbf{q}^T$. The identity obtained by comparing the terms with v^3 takes the form

$$n v_3 v_2 (\delta q_3 q_1^{-1} + f_{33}) \equiv \langle \boldsymbol{\nu}, K\mathbf{m}_3, H_2'\boldsymbol{\nu} \rangle$$

which is equivalent to the following system of conditions for the elements χ_{ij}' of the operator H_2' in Γ

$$\chi_{32}' = \chi_{21}' = 0, \quad k_{13}\chi_{13}' = k_{33}(\chi_{11}' - \chi_{22}')$$

$$n(\delta q_3 q_1^{-1} + f_{33}) = k_{33}\chi_{13}' + k_{13}(\chi_{22}' - \chi_{33}')$$

These conditions yield the following representation for the operator H_2'

$$H_2' = \chi_{22}' E + \chi_1 \mathbf{m}_3 \mathbf{m}_3^T + \chi_2 (K\mathbf{m}_3)(K\mathbf{m}_3)^T$$

$$\chi_1 = -k_{13}^{-1} n (\delta q_3 q_1^{-1} + f_{33}), \quad \chi_2 = \chi_{13}' (k_{33} k_{11})^{-1}$$

Separating out the terms with zv^2 in identity (3.7) and taking the last representation for C into consideration, we obtain

$$\begin{aligned} &(\mathbf{q}, \boldsymbol{\nu})[(\mathbf{q}, \boldsymbol{\nu}) k_{13} z_2 - n(k_{33} - k_{22}) z_2 v_3 + n k_{13} z_1 v_2 + \langle \mathbf{z}, \boldsymbol{\nu}, F\mathbf{m}_3 \rangle + \delta q_1^{-1} \langle \mathbf{z}, \boldsymbol{\nu}, \mathbf{q} \rangle] \\ &- n v_3 \langle \mathbf{m}_1, \boldsymbol{\nu}, F\mathbf{z} \rangle + \langle \boldsymbol{\nu}, K\mathbf{z}, H_2'\boldsymbol{\nu} \rangle = 0 \end{aligned} \tag{3.9}$$

Setting $\mathbf{z} = \mathbf{m}_2$, $\boldsymbol{\nu} = \mathbf{m}_1$, we obtain, taking relation (3.4) into account,

$$\chi_1 - 2k_{11}k_{33}\chi_2 = 2q_0^2 k_{13}^2 \tag{3.10}$$

Setting $\mathbf{z} = \mathbf{m}_2$, $\boldsymbol{\nu} = \mathbf{m}_3$ and taking equalities (1.7) and (1.10) into consideration, we obtain

$$k_{11}k_{33}\chi_2 + q_0^2(2k_{33}^2 + 2k_{13}^2 - k_{11}k_{33}) = 0 \tag{3.11}$$

Setting $\mathbf{z} = \mathbf{m}_1$, $\boldsymbol{\nu} = \nu_1\mathbf{m}_1 = \nu_2\mathbf{m}_2$ in relation (3.9), we obtain

$$-\chi_1 + 2\chi_2(k_{11}k_{33} - k_{13}^2) + 4q_0^2k_{13}^2 = 0 \tag{3.12}$$

It follows from equalities (3.10) and (3.12) that $\chi_2 = q_0^2$, but then (3.11) will hold only if $k_{33}^2 = k_{13}^2 = 0$, which is impossible.

Thus, no QI exists with $n \neq 0$, but then, by representation (3.4), $H_3 = 0$, $\mathbf{q} = 0$. Taking relations (1.7), (1.10) and (3.5) into consideration, we obtain

$$f_{11} = f_{22} = f_{31} = 0, \quad \delta = -f_{13} \tag{3.13}$$

These conditions for $F = \mathrm{BK}$ yield

$$\beta_{13} = -\delta k_{11}d^{-1} = -f_{33}k_{13}d^{-1}, \quad f_{33} = k_{11}k_{13}^{-1}\delta$$

(β_{ij} are the elements of the operator B in Γ).

Together with relations (3.13), this gives a representation of F in the form (3.8). Since $K\mathbf{b} = d\mathbf{m}_3$, it follows from this representation that we can express the operator B in the form (1.16). Representation (1.16) for the operator C when $\mathbf{q} = 0$ is obtained from (1.4).

Note that, since $\mathbf{b} - dA\mathbf{m}_3$, the first expression in (1.16) implies

$$B = \delta dk_{13}^{-1}A\mathbf{m}_3\mathbf{m}_3^TA$$

In identity (3.7), the sums of the terms with z^3, $z^2\nu$, ν^3, z^2 vanish. The sum of the terms with z^1: λ^1 $\langle \mathbf{m}^1, \alpha\mathbf{m}^3 = \lambda, K\mathbf{z}\rangle = \lambda_1\alpha\langle \mathbf{m}_1, \mathbf{z}, K\mathbf{m}_3\rangle + \alpha\langle \mathbf{z}, \alpha\mathbf{m}_3 + \lambda, K\mathbf{m}_3\rangle$ is equal to $z_2(\alpha + \lambda_3)(k_{11}\lambda_1 - \alpha k_{13})$, and it vanishes because of condition (1.5). We write out the remaining terms with $z\nu^2$, $z\nu$, ν^2, ν:

$$c\nu_3\langle \mathbf{z}, \boldsymbol{\nu}, \mathbf{m}_3\rangle + \langle \boldsymbol{\nu}, K\mathbf{z}, H_2'\boldsymbol{\nu}\rangle = 0, \quad c = c_{33} - c_{22} \tag{3.14}$$

$$\langle \mathbf{z}, \boldsymbol{\nu}, \mathbf{s}\rangle + \alpha\delta k_{13}^{-1}\langle \mathbf{z}, \boldsymbol{\nu}, \mathbf{b}\rangle - \langle \boldsymbol{\nu}, K\mathbf{z}, \mathbf{h}_2'\rangle = 0 \tag{3.15}$$

$$c\lambda_1\nu_2\nu_3 + \alpha\langle \boldsymbol{\nu}, K\mathbf{p}, H_2'\boldsymbol{\nu}\rangle \equiv 0 \tag{3.16}$$

$$\lambda_1\langle \mathbf{m}_1, \boldsymbol{\nu}, \mathbf{s}\rangle + \alpha\lambda_1\delta k_{13}^{-1}\langle \mathbf{m}_1, \boldsymbol{\nu}, \mathbf{b}\rangle - \alpha\langle \boldsymbol{\nu}, K\mathbf{m}_3, \mathbf{h}_2'\rangle = 0 \tag{3.17}$$

Proposition 11. In a QI other than (1.2), the operator H_2 has the form

$$H_2 = \chi_3E + \chi ck_{11}^{-1}\mathbf{m}_3\mathbf{m}_3^T \tag{3.18}$$

Proof. Analysis of condition (3.14) shows that it holds only if $H_2' = H_2/\chi$, where H_2 has the form (3.18). It is not difficult to verify, using condition (1.5), that if H_2 has the form (3.18), condition (3,16) is satisfied.

Proposition 12. In a QI other than (1.2), the parameter $\mathbf{h}_2$ has the form

$$\mathbf{h}_2 = -\chi(k_{11}k_{13})^{-1}(\mathbf{s}, K\mathbf{m}_1)\mathbf{m}_3 \tag{3.19}$$

Proof. Setting $\mathbf{z} = \mathbf{m}_1$, $\boldsymbol{\nu} = \mathbf{m}_3$ in condition (3.15), we obtain $(\mathbf{h}_2', \mathbf{m}_2) = 0$. Setting $\mathbf{z} = \mathbf{m}^2$, $\boldsymbol{\nu} = \mathbf{m}_3$ in the same condition and taking (1.6) into consideration, we obtain $(\mathbf{h}_2', \mathbf{m}_1) = 0$. Consequently, $\mathbf{h}_2'$, $= h\mathbf{m}_3$, and condition (3.15) becomes

$$\langle \mathbf{z}, \boldsymbol{\nu}, \mathbf{s}\rangle + \alpha\delta k_{13}^{-1}\langle \mathbf{z}, \boldsymbol{\nu}, \mathbf{b}\rangle = h\langle \boldsymbol{\nu}, K\mathbf{z}, \mathbf{m}_3\rangle$$

Setting $\mathbf{z} = \mathbf{m}_1$ and $\mathbf{z} = \mathbf{m}_2$ in this condition, we see that it holds if and only if

$$h = -(k_{11}k_{13})^{-1}(k_{13}s_3 + k_{11}s_1)$$

This enables us to write $\mathbf{h}_2$ in the form (3.19).

By virtue of relations (3.19), (1.5) and (1.6) condition (3.17) holds.

The combination of all these necessary conditions for a QI to exist is sufficient for identity (3.1) to hold. Using the representations established for the operators H_1, H_2 and H_3 and the parameters $\mathbf{h}_1$ and $\mathbf{h}_2$, we can write the QI (1.14) in the form (1.5).

4. REDUCING THE ORDER OF THE SYSTEM

Given the existence of a QI (1.14), we can write system (1.13) in the form

$$\begin{aligned} &\mathbf{z}^{\cdot} = (\mathbf{z} + \alpha\mathbf{m}_3 + \boldsymbol{\lambda}) \times K(\mathbf{z} + \alpha\mathbf{m}_3) + (c\nu_3 - c_2)\boldsymbol{\nu} \times \mathbf{m}_3 \\ &\boldsymbol{\nu}^{\cdot} = \boldsymbol{\nu} \times K(\mathbf{z} + \alpha\mathbf{m}_3), \quad c_2 = k_{13}^{-1}(\mathbf{s}, K\mathbf{m}_1) \end{aligned} \tag{4.1}$$

Setting $\mathbf{y} = \mathbf{z} + \lambda$, we express the system in terms of the basis Γ:

$$\begin{aligned} &y_1^{\cdot} = y_2(k_{13}y_1 + c_3) + (c\nu_3 - c_2)\nu_2 \\ &y_2^{\cdot} = -y_2(k_{13}y_1 + c_3) - (c\nu_3 - c_2)\nu_1 \\ &\nu_1^{\cdot} = \nu_2(k_{13}y_1 + c_4) - k_{11}\nu_3 y_2 \\ &\nu_2^{\cdot} = -\nu_2(k_{13}y_1 + c_4) + k_{11}\nu_3 y_1 \\ &\nu_3^{\cdot} = k_{11}(\nu_1 y_2 - \nu_2 y_1), \quad c_3 = c_4 - k_{11}(\alpha + \lambda_3), \quad c_4 = k_{11}^{-1}\alpha d \end{aligned} \tag{4.2}$$

Integrals (1.2) and (1.15) become

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1, \quad y_1\nu_1 + y_2\nu_2 + (\alpha + \lambda_3)\nu_3 = h, \quad c\nu_3^2 - 2c_2\nu_3 + k_{11}y^2 = g \tag{4.3}$$

Put

$$y_1 = y\cos\xi, \quad y_2 = y\sin\xi \tag{4.4}$$

The first two integrals of (4.3) give

$$\begin{aligned} &\nu_1 = b_2\cos\xi + b_3\sin\xi, \quad \nu_2 = b_2\sin\xi - b_3\cos\xi \\ &b_1 = (1 - \nu_3^2)^{1/2}, \quad b_2 = [h - (\alpha + \lambda_3)\nu_3]y^{-1}, \quad b_3 = \pm(b_1^2 - b_2^2)^{1/2} \end{aligned} \tag{4.5}$$

It follows from system (4.2) that

$$(y^2)^{\cdot} = 2(c\nu_3 - c_2)(\nu_2 y_1 - \nu_1 y_2)$$

which gives $(y^2)^{\cdot} = -2(c\nu_3 - c_2)b_3 y$ and, if $c\nu_3 + c_2$, then, taking the third of the integrals of (4.3) into account, we obtain an equation for ν_3 which is integrable in terms of elliptic functions

$$\nu_3^{\cdot} = \pm\{k_{11}(g - c\nu_3^2 + 2c_2\nu_3)(1 - \nu_3^2) - k_{11}^2[h - (\alpha + \lambda_3)\nu_3]^2\}^{1/2} \tag{4.6}$$

The solution of system (4.2) reduces to integrating the equation

$$\begin{aligned} &\xi^{\cdot} = f_1(t)\cos\xi + f_2(t) \\ &f_1(t) = -k_{13}y, \quad f_2(t) = -c_2 - (c\nu_3 - c_2)[h - (\alpha + \lambda_3)\nu_3]y^{-2} \end{aligned} \tag{4.7}$$

The problem of determining, e.g. the Euler angles of the basis Γ may also be reduced to solving a first-order equation.

If $c\nu_3 \equiv c_2$, then y = const and y_1, y_2 may be expressed in terms of elementary functions.

If $k_{13}y \geqslant |c_3|\ \xi(t)$, then as $t \to +\infty$, $\xi(t)$ tends to a limit ξ^* given by the condition

$$\cos\xi^* = -c_3(k_{13}y)^{-1}$$

If $k_{13}y < |c_3|$, we obtain a periodic solution, given for $c \neq 0$ by formulae (4.4) and (4.5) where, in accordance with Eq. (4.7),

$$\cos\xi = (b - a\cos\tau)(b\cos\tau - a)^{-1}$$

$$\sin\xi = (a^2 - b^2)^{1/2}\sin\tau(b\cos\tau - a)^{-1}$$

$$a = c_3, \quad b = k_{13}y, \quad \tau = (a^2 - b^2)^{1/2}t + c_5$$

Since $v_3 = c_2c^{-1}$ = const, the axis $\mathbf{m}_3$ attached to the body precesses about the vertical.

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